ERGODIC AUTOMORPHISMS WITH SIMPLE SPECTRUM CHARACTERIZED BY FAST CORRELATION DECAY

A. A. PRIKHOD'KO

Abstract. The existence of measure preserving invertible transformations T on a Borel probability space $(X, \mathscr{B}, \mu)$ with simple spectrum is established possessing the following rate of correlation decay for a dense family of functions $f \in L^2(X, \mu)$:

$$\forall\, \varepsilon > 0 \qquad \langle f(T^k x), f(x)\rangle = O(|k|^{-1/2+\varepsilon}).$$

According to identity $\langle f(T^k x), f(x)\rangle = \hat{\sigma}_f(k)$, where σ_f denotes the spectral measure associated with f, the rate of decay of the Fourier coefficients $\hat{\sigma}_f(k)$, observed for the class of transformations introduced in the paper, is the maximal possible for singular Borel measures on $[0, 1]$.

This note summarizes the results of the paper arXiv:1008.4301.

Let us consider an invertible measure preserving transformation T on a Borel probability space $(X, \mathscr{B}, \mu)$ and recall a question which is well-known in the spectral theory of ergodic dynamical systems and goes back to Banach: *Does there exist a transformation with invariant probability measure having Lebesgue spectrum of multiplicity one?*

Ulam [1] (ch. VI, § 6) states this problem in the following way.

Does there exist a function $f \in L^2(X, \mu)$ and a measure preserving invertible transformation $T\colon X \to X$ (an automorphism), such that the sequence of functions $\{f(T^k x)\colon k \in \mathbb{Z}\}$ is a complete orthogonal system in the Hilbert space $L^2(X, \mathscr{B}, \mu)$?

One can easily construct an example of dynamical system of such kind on the space with *infinite* measure. Let us consider the set of integers $\mathbb{Z}$ as a phase space X with the standard counting measure $\nu(\{j\}) \equiv 1$, and let us define $T\colon\ j \mapsto j + 1$. Then the functions $T^k \boldsymbol{\delta}_0$ constitute a basis in $L^2(\mathbb{Z}, \nu)$, where $\boldsymbol{\delta}_0(j) = 1$ if $j = 0$ and $\boldsymbol{\delta}_0(j) = 0$ if $j \neq 0$.

In the class of finite measure preserving transformations the hypothesis of Banach is still open. Kirillov [2] states a generalized hypothesis for general Abelian group actions on a space with a finite probability measure.

We concentrate on the case $G = \mathbb{Z}$. For a survey of constructions and results in the spectral theory of ergodic dynamical systems the reader can refer to [3] and [4]. The dual group $\hat{\mathbb{Z}}$ is isomorphic to the unit circle S^1 in the complex plane and the Fourier coefficients of any Borel probability measure σ on S^1 are recovered from the identity

$$\hat{\sigma}(k) = \int_{S^1} z^k \, d\sigma, \qquad k \in \mathbb{Z}.$$

[0] *Submitted to Russian Math. Notes (2012)*

The work is supported by RFFI grant No. 11-01-00759-a and the grant "Leading Russian scientific schools" No. NSh-5998.2012.1.

Definition 1. Le $\kappa(\sigma)$ denote the liminf of the values $\alpha \in \mathbb{R}$ satisfying the estimate $\hat\sigma(k) = O(|k|^{\alpha+\varepsilon})$ for any $\varepsilon > 0$.

Since the group S^1 is compact then any σ satisfying $\hat\sigma(k) = O(|k|^{-1/2-c})$ for some $c > 0$ is absolutely continuous with respect to the normalized Lebesgue measure λ on S^1. In particular, $\sigma = p(x)\lambda$, where $p(x) \in L^1(S^1, \lambda)$. Thus, any measure on $[0, 1]$ with $\kappa(\sigma) < -1/2$ is absolutely continuous.

Given a function $f \in L^2(X, \mathscr{B}, \mu)$ consider a sequence of auto-correlations

$$R_f(k) = \langle f(T^k x), f(x)\rangle,$$

and the spectral measure σ_f associated with f and given by $\hat\sigma_f(k) = R_f(k)$. Whenever $\kappa(\sigma_{f_j}) < -1/2$ holds for a dense family $\{f_j\} \in L^2(X) = L^2(X, \mathscr{B}, \mu)$, the spectrum of T is absolutely continuous. We will prove that an extreme value $\kappa(\sigma_f) = -1/2$ (on a dense set of functions) is achieved in the class of ergodic transformations with simple spectrum.

Theorem 1. *There exists an automorphism T on a Borel space $(X, \mathscr{B}, \mu)$ with simple spectrum such that $\kappa(\sigma_f) \le -1/2$ for a dense set of functions $f \in L^2(X)$.*

Let us define

$$\kappa(T) = \inf_{\mathscr{F} \text{ dense in } L^2(X)} \ \sup_{f \,\in\, \mathscr{F}} \kappa(\sigma_f).$$

Thus, theorem 1 states the existence of T with simple spectrums and $\kappa(T) \le -1/2$. Observe that $\kappa(T) = -\infty$ for any T with Lebesgue spectrum.

Theorem 2. *Let T be an automorphism satisfying the equality $\kappa(T) = -1/2$, and let σ be the maximal spectral type of T. Then $\sigma * \sigma \ll \lambda$, and, furthermore, the spectrum of T either contains an absolutely continuous component or is purely singular, and for any spectral measure σ_f we have $\kappa(\sigma_f) = -1/2$.*

Throughout this paper we call singular Borel measures satisfying $\kappa(\sigma_f) = -1/2$ *Salem–Schaeffer measures*. This class of probability distributions were studied in the works of Schaeffer, Salem, Sigmund, Ivashev-Musatov et al. (see [5, 6, 7]).

The main idea of this work is to show that Salem–Schaeffer measures are found among spectral measures of ergodic dynamical systems. We propose a construction of a class of automorphisms that will serve an example of such kind.

Definition 2. (*Symbolic construction*) Let $\mathbb{A}$ be a finite alphabet and let w_0 be a finite word in $\mathbb{A}$ containing at least two different letters. Denote by $\rho_\alpha(w)$ the *cyclic shift* of w to the left:

$$\rho_1(au) = ua, \qquad \rho_\alpha(u) = \rho_1^\alpha(u), \qquad a \in \mathbb{A} \text{ — a letter}, \quad u \text{ — a word}.$$

Let us construct the sequence of words w_n applying the next rule:

$$w_{n+1} = \rho_{\alpha_{n,0}}(w_n)\, \rho_{\alpha_{n,1}}(w_n)\, \ldots\, \rho_{\alpha_{n,q_n-1}}(w_n). \tag{1}$$

Here the sequences $q_n \in \mathbb{N}$ and $\alpha_{n,j} \in \mathbb{Z}/h_n\mathbb{Z}$, $h_n = |w_n|$ serve as parameters of the construction, and $|w|$ denotes the length of the word w. Without loss of generality, one can assume the first entry of w_n inside the bigger word w_{n+1} is not touched, $\alpha_{n,0} \equiv 0$. Then every w_n is a prefix of the successor word w_{n+1}, and we can define a unique infinite word w_∞ expanding every word w_n to the right.

Further, applying a standard procedure let us define the minimal compact subset $K \subset \mathbb{A}^{\mathbb{Z}}$ containing all the shifts of the word w_∞. The left shift transformation $T\colon\ (x_j) \mapsto (x_{j+1})$ provides a topological dynamical system acting on the set K.

Futher, let us endow the set K with a natural Borel measure μ invariant under T. We define the probability $\mu(u)$ of the word u to be the asymptotic frequency of observing u as subword in w_∞.

The construction and ergodic properties of the ergodic system $(T, K, \mathscr{B}, \mu)$ are discussed in details in [8]. The complexity characteristics of the topological system (T, K), as well as the infinite word w_∞ are studied in [9]. An infinite sequence of concateneted cyclic shifts of a fixed word was previously studied in the theory of recursive functions [10]. Setting $h_1 = 2$, $q_n \equiv 2$, $\rho_{n,0} = 0$, $\rho_{n,1} = h_n/2$, we see that the classical *Morse automorphism* (see [11]) is includeded in the class definied above. It can be also shown that the constructed systems possess adic representation.

Definition 3. (*Algebraic construction*) Let h_n be a sequence of positive integers such that $h_{n+1} = q_n h_n$, $q_n \in \mathbb{N}$, $q_n \neq 1$. Consider then a sequence of embedded lattices $\Gamma_n = h_n\mathbb{Z}$, where $\Gamma_{n+1} \subset \Gamma_n$, and the corresponding homogeneous spaces $M_n = \mathbb{Z}/\Gamma_n = \mathbb{Z}_{h_n}$.

Let us fix projections $\phi_n\colon\ M_{n+1} \to M_n$ definied by

$$\phi_n\colon\ jh_n + k \mapsto k + \alpha_{n,j} \pmod{h_n}, \qquad 0 \le k < h_n, \quad j = 0, 1, \dots, q_n.$$

Evidently, ϕ_n preserve normalized Haar measures μ_n on the Borel spaces M_n. Define the phase space X as inverse limit of spaces $(M_n, \mathscr{B}_n, \mu_n)$, namely, set

$$X = \bigl\{x = (x_1, x_2, \dots, x_n, \dots)\colon\ \phi_n(x_{n+1}) = x_n\bigr\}.$$

The measures μ_n become Borel measures μ on X. Let us define the transformation T on the space $(X, \mathscr{B}, \mu)$ as follows. Any projection ϕ_n almost commutes with the shift transformation on M_n,

$$\mu\bigl\{x\colon\ \phi_n(S_{n+1}(x_{n+1})) \neq S_n(\phi_n(x_{n+1}))\bigr\} \le h_n^{-1},$$

hence, applying Borrel–Cantelli lemma, we see that μ-almost surely the equality $\phi_n(S_{n+1}(x_{n+1})) = S_n(\phi_n(x_{n+1}))$ holds for $n \ge n^*(x)$, where $n^*(x)$ is a measurable function. Set

$$(Tx)_n = x_n + 1, \quad n \ge n^*(x), \quad \text{and} \quad (Tx)_n = \phi_n(Tx_{n+1}), \quad n < n^*(x).$$

Lemma 1 (see [8])**.** *The map T is a measure preserving invertible transformation of the probability space $(X, \mathscr{B}, \mu)$.*

The equivalence of the two constructions introduced above is verified via coding T-orbits by words w_n induced by functions $M_{n_0} \to \mathbb{A}$ for some n_0.

In order to prove theorem 1 we consider a certain stochastic family of dynamical systems $(T, X, \mathscr{B}, \mu)$ constructed above, depending on random parameters. Then we show that T has simple spectrum and the required rate of correlation decay almost surely with respect to the probability on the set of parameters.

Theorem 3. *There exists a sequence $q_n \in \mathbb{N}$ such that the transformation T defined above with $\alpha_{n,j}$ independent and uniformly distributed on M_n has simple spectrum and satisfy the inequality $\kappa(T) \le -1/2$.*

Proof. The detailed proof of the simplicity of spectrum is given in [8]. It is based on the following lemma.

Lemma 2 (see [12]). *Let U be a unitary operator in a separable Hilbert space H, let σ be the measure of maximal spectral type and let $\mathcal{M}(z)$ denote the multiplicity function of the operator U. If $\mathcal{M}(z) \geq m$ on a set of positive σ-measure then there exist m orthogonal elements of unit length $f_1, \ldots, f_m$ such that for any cyclic space $Z \subset H$ (with respect to U) and for any m elements $g_1, \ldots, g_m \in Z$ of the same length $\|g_i\| = a$ the inequality inequality holds*

$$\sum_{i=1}^{m} \|f_i - g_i\|^2 \geq m(1 + a^2 - 2a/\sqrt{m}).$$

In order to prove the second statement of the theorem it is enough to estimate the decay of correlations for $\mathscr{B}_{n_0}$-measurable (cylinfric) functions $f(x)$ which are dense in $L^2(X)$. Any such function $f(x)$ can be represented in the form $f(x) = f_{n_0}(x_{n_0})$, where $f_{n_0}\colon M_{n_0} \to \mathbb{C}$ and x_n is the n-th coordinate of a point x. Then for any $n > n_0$

$$f(x) = f_n(x_n), \quad \text{where} \quad f_{n+1}(x_{n+1}) = f_n(\phi_n(x_{n+1})).$$

Given a function $f(z)$ with zero mean define the *cyclic correlations*

$$R_n^\circ(t) = \int_{M_n} f_n(j+t)\, \overline{f_n(j)}\, d\mu_n(j),$$

where $j+t$ is the sum in the group M_n, i.e. $j+t \pmod{h_n}$. Taking into account the conditions on the distribution of the randomparameters $\alpha_{n,j}$ it can be easily shown that μ-a.s. $R_n^\circ(t) \to R_f(t)$ for any t, hence, the distributions $\hat{R^\circ}_n$ converges weakly to the spectral measure σ_f (but we need only the first convergence). Now let us consider (the most important) case, when $t = sh_n$, $s \neq 0$. For this special value of the argument t the following recurrent identity holds

$$R_{n+1}^\circ(t) = \frac{1}{q_n} \sum_{k=0}^{a_n - 1} R_n^\circ(\alpha_{n,k+s} - \alpha_{n,k}).$$

Applying expectation operator with respect to the probability on the parameters' space, we obtain $\mathsf{E}\, R_{n+1}^\circ(t) = \mathsf{E}\, R_n^\circ(\alpha_{n,k+s} - \alpha_{n,k}) = 0$ and

$$\mathsf{E}\, |R_{n+1}^\circ(t)|^2 = \frac{1}{q_n^2}\, \mathsf{E} \sum_{k,\ell=0}^{q_n - 1} R_n^\circ(\alpha_{n,k+s} - \alpha_{n,k})\, \overline{R_n^\circ(\alpha_{n,\ell} - \alpha_{n,\ell+s})}.$$

Observe that all the terms in the above sum are zero except the terms such that $\{k, k+s\} = \{\ell, \ell+s\}$. If h_n is odd these are always the terms with $k = \ell$, and if n is even we should also count the terms given by $k+s = \ell$, and $\ell + s = k$. Clearly, the latter contributes $O(q_n^{-1})$ to the sum over all s, so without loss of generality we can assume that all h_n are odd. We have then

$$\mathsf{E}\, |R_{n+1}^\circ(t)|^2 = \frac{1}{q_n}\, \mathsf{E}\, |R_n^\circ(\alpha_{n,k+s} - \alpha_{n,k})|^2,$$

hence, $\mathsf{E}\, |R_{n+1}^\circ(t)|^2 = h_{n+1}^{-1}\, \mathsf{E}\, \|R_n^\circ\|^2$, where $\|\cdot\|$ is a standard form in $L^2(\mathbb{Z})$, and the function R_n° is restricted to $[0, h_n - 1]$. The same arguments slightly modified lead the equality $\mathsf{E}\, |R_{n+1}^\circ(t)|^2 = h_{n+1}^{-1}\, \mathsf{E}\, \|R_n^\circ\|^2$ for any $t \in (h_n, h_{n+1})$. Thus, one can see that

$$\mathsf{E}\, \|R_{n+1}^\circ\|^2 \leq 2 \cdot \mathsf{E}\, \|R_n^\circ\|^2.$$

It follows that $R_{n+1}^\circ(t) = O(|t|^{-1/2+\varepsilon})$ for any $\varepsilon > 0$ the second statement of the theorem is verified. $\square$

Definition 4. We say that an automorphism T of a Borel probability space $(X, \mathscr{B}, \mu)$ admits *approximation of type* $\mathcal{I}$ if for any finite partition $\mathscr{P}$ and any $\varepsilon > 0$ there exists a subset $\Omega_\varepsilon \subset X$ of the measure $1 - \varepsilon$ and a word W_ε such that for all $x \in \Omega_\varepsilon$ the infinite word generated by $\mathscr{P}$-coding of the x-orbit is ε-covered by a sequence of words $\tilde{W}_j$ which are $\bar{d}$-ε-close to cyclic shifts $\rho_{\alpha_j}(W_\varepsilon)$ of the word W_ε.

This property is a metric invariant. In particular, the class of maps satisfying type $\mathcal{I}$ approximation includes rank one transformations. Clearly, the main construction of this paper generates transformations of type $\mathcal{I}$.

Hypothesis 1. Let T be an automorphism constructed according to definition 2 (or definition 3). Consider an arbitrary $\mathscr{B}_n$-measurable function f with zero mean. Then $\kappa(\sigma_f) \geq -1/2$.

Hypothesis 2. Assume that an automorphism T admits approximation of type $\mathcal{I}$, and suppose that ξ_n are finite partitions generating, for any fixed finite partition $\mathscr{P}$, approximating sequence of words W_{ε_n} with $\varepsilon_n \to 0$. Then

$$\liminf_{k\to\infty} \inf_{f \text{ is } \xi_n\text{-measurable}} \kappa(\sigma_f) \geq -1/2$$

The author thanks A. M. Vershik, V. V. Ryzhikov, J.-P. Thouvenot, E. Janvresse, T. de la Rue, K. Petersen, B. Weiss, and the participants of seminar “Probability theory and statistical phyics” for discussions and remarks.

References

1. S.M. Ulam. A collection of mathematical problems. Interscience Tracts in Pure and Applied Mathematics, no. 8. New York: Interscience , 1960
2. A.A. Kirillov. Dynamical systems, factors and representations of groups. *Russian Math. Surveys,* **22** (1967), 63-75
3. A. Katok, J.-P. Thouvenot. Spectral theory and combinatorial constructions. Handbook on dynamical systems, Volume 1, Part B. Amsterdam: Elsevier , 2006
4. M. Lemanczyk. Spectral Theory of Dynamical Systems. Encyclopedia of Complexity and System Science: Springer-Verlag , 2009, 8554-8575
5. A. C. Schaeffer. The Fourier-Stieltjes coefficients of a function of bounded variation. *Amer. J. Math.,* **61** (1939), 934-940
6. R. Salem. On some singular monotonic functions which are strictly increasing. *Trans. Amer. Soc. Math.,* **53** , 1943, 427-439
7. O.S. Ivasev-Musatov. On Fourier-Stieltjes coefficients of singular functions. *Izv. Akad. Nauk SSSR Ser. Mat.,* **20** (1956), 179-196
8. A. A. Prikhod'ko. On ergodic properties of iceberg transformations. I: Approximation and spectral multiplicity. *arXiv:1008.4301* (2010), 1-23
9. A. A. Prikhod'ko. Lower bounds for symbolic complexity of iceberg dynamical systems. *arXiv:1201.5757* (2011), 1-21
10. Ebbinghaus H.D., Jacobs K., Mahn F.K., Hermes H.. Selecta mathematica II: Turing-Maschinen und berechenbare Funktionen: Springer-Verlag , 1970
11. A. M. Vershik, B. Solomyak. The Adic Realization of the Morse Transformation and the Extension of its Action on the Solenoid. *Preprint ESI 2068, Vienna* (2008), 1-18
12. I.P. Kornfel'd and Ya.G. Sinaĭand S.V. Fomin. Ergodic theory. Berlin – Heidelberg – New York: Springer-Verlag , 1982

Lomonosov Moscow State University, Moscow
E-mail address: sasha.prihodko@gmail.com